Non-perturbative renormalisation using dimensional regularisation: applications to the $\epsilon$ expansion



1984 J. Phys. A: Math. Gen. 17 1861

(http://iopscience.iop.org/0305-4470/17/9/021)

J. Phys. A: Math. Gen. 17 (1984) 1861–1876. Printed in Great Britain

# Non-perturbative renormalisation using dimensional regularisation: applications to the $\varepsilon$ expansion

A J McKane†, D J Wallace† and O F de Alcantara Bonfim‡

† Department of Physics, University of Edinburgh, Mayfield Road, Edinburgh EH9 3JZ, Scotland

‡ Departamento de Fisica, Universidade Federal de Pernambuco, 50.000 Recife PE, Brazil

Received 4 January 1984

**Abstract.** We give a prescription for the one-loop renormalisation of the imaginary parts of vertex functions in $g\phi^4$, which are generated when $g<0$. Within the dimensional regularisation scheme this involves another, non-perturbative, renormalisation over and above the usual one-loop perturbative renormalisation. We use an extended minimal subtraction scheme so that the resulting renormalisation group functions have a simple dependence on $\varepsilon = 4-d$. In contrast to the usual case, however, both the $\beta$ function and the renormalised coupling $g_R$ are in general complex for real $g$, and the fixed point for $\varepsilon, g_R < 0$ is non-perturbative in $\varepsilon$. This fixed point determines the imaginary parts of the critical exponents which are generated when $\varepsilon < 0$, and allows us to determine the high-order behaviour of the perturbation series in $\varepsilon$ for these exponents. The generalisation of these ideas to the O($n$) symmetric $g(\phi^2)^2$ model is also given.

## 1. Introduction

The method of calculating critical exponents using the field theoretic formulation of the renormalisation group is now a remarkably accurate one. This is largely a result of the increased understanding of the longer perturbation series in field theory achieved in recent years. In particular, Lipatov (1977a, b) and Brézin *et al* (1977a) have given a method for calculating the $K$th-order term in the perturbation series, for $K$ large. This information has been used as the basis of various resummation methods, which lead to more accurate critical exponents (Le Guillou and Zinn-Justin 1977, 1980).

The determination of the large-order behaviour of the perturbation series in field theory begins with the observation that the imaginary part of the Green functions, generated when no stable ground state of the system exists (for instance, $g\phi^4$ when $g<0$), can be calculated by studying the classical solutions to the field equations (the instantons) of the theory (Zinn-Justin 1982). These imaginary parts are exponentially small in the coupling constant, since they describe a tunnelling process. Typically,

$$\text{Im}\, G = C_1|g|^{-\alpha} \exp(-C_0/|g|)[1 + C_2|g| + \text{O}(g^2)] \tag{1.1}$$

where $G$ is a Green function and $\alpha$, $C_0$, $C_1$, $C_2, \ldots$ depend on the theory and Green function under study. Whereas $C_0$ can be found by a purely classical calculation (that is, a zero-loop calculation), the $C_l$, $l = 1, 2, \ldots$, come from performing an $l$-loop calculation about the instanton solution. Having obtained Im $G$ for an unphysical value of the coupling constant, the coefficient of $g^K$ in Re $G$ for a physical value of the coupling constant (for instance $g>0$ in $g\phi^4$) is obtained by writing a dispersion relation in $g$. This coefficient, $G_K$, has the form

$$G_K = \bar{C}_1 K! K^{\alpha-1}(\bar{C}_0)^K[1+\bar{C}_2/K+O(K^{-2})], \qquad K \text{ large}, \tag{1.2}$$

where the $\bar{C}_l$ are directly related to the $C_l$ of (1.1). Since $C_0$ and $\alpha$ are not difficult to obtain, the leading behaviour of the perturbation series at high order is easily found.

The same cannot be said of the corrections to this leading behaviour, that is, the overall scale $\bar{C}_1$ and the series in $K^{-1}$: while the calculation of $C_1, C_2, \dots$ is in principle relatively straightforward, in practice it is quite complicated. In quantum mechanics (one-dimensional field theory) where the technical problems are minimised, $C_1$ has been calculated for a large variety of systems (Brézin *et al* 1977b) and $C_2$ for only one or two simple cases (see e.g. Collins and Soper 1978, Lowe and Stone 1978).

The most difficult new problem encountered in moving from quantum mechanics to higher-dimensional field theories is that of renormalisation. Yet it is the development of a systematic renormalisation scheme, for the imaginary part as well as the real part of the vertex functions, that is needed before the constants $C_l$, $l=1,2,\dots$, can be calculated for the renormalisation group functions and thus for the critical exponents. To date only the classical numbers $C_0$ and $\alpha$ are known for exponents in the $\varepsilon$ expansion. The purpose of this paper is to introduce such a scheme for $g\phi^4$ theory in $4-\varepsilon$ dimensions and to implement it to lowest order. The renormalisation constants are calculated using an extended minimal subtraction scheme and have a non-perturbative imaginary part for $g<0$, in addition to the usual real part. This enables us to find the imaginary part of the renormalisation group functions for $g<0$; they are of the form (1.1) with $C_1$ being a finite calculable number. We can then go on to show that the critical exponents calculated in an $\varepsilon$ expansion themselves have an imaginary, non-perturbative part of the form (1.1) but now in the variable $\varepsilon$ and only for $\varepsilon<0$. For $\varepsilon>0$ they are real and perturbative in $\varepsilon$. A dispersion relation in $\varepsilon$ then gives the constant $\bar{C}_1$ for the various exponents. This is a universal number, unlike the overall constants for vertex functions and renormalisation group functions, which depend on the renormalisation scheme.

The plan of the paper is as follows. In § 2 we introduce the renormalisation scheme for the full (real and imaginary) vertex functions of $g\phi^4$ theory ($g<0$) and calculate the renormalisation constants to one loop. In § 3 these renormalisation constants are used to find the imaginary parts of the renormalisation group functions for $g<0$ and thus the imaginary parts of the critical exponents for $\varepsilon=4-d<0$. The behaviour of the coefficient of $\varepsilon^K$ for $K$ large is found for the exponents $\eta$, $\nu^{-1}$ and $\omega$ in § 4, and in particular the universal numbers $\bar{C}_1$ are given for these three exponents. The generalisations of these results to the O($n$) symmetric $g(\phi^2)^2$ case are also presented. The computational details of the general $n$ case are given in an appendix.

## 2. Renormalisation

The imaginary part of the vertex functions for $g\phi^4$ theory ($g<0$) were first calculated by Lipatov (1977a, b) and Brézin *et al* (1977a), both of whom used a momentum space cut-off as the ultraviolet regulator. When studying the nature of the $\varepsilon$ expansion it is more natural to use dimensional regularisation, and so our starting point will be later work which used this form of regularisation (McKane and Wallace 1978, Drummond and Shore 1979). Indeed, this was the chief motivation for the work described in McKane and Wallace (1978) (referred to as MW from now on).

Let us for illustration restrict ourselves in the first instance to the four-point vertex function. A one-loop calculation about the instanton gives the following result ((37)–(39) of MW):

$$\operatorname*{Im}_{(\arg g=\pi)}\Gamma_{\rm b}^{(4)}(q_i)=-C_{\rm b}\int_0^\infty \frac{{\rm d}\lambda}{\lambda}\lambda^{\varepsilon}\left(-\frac{\lambda^{\varepsilon}A}{g}\right)^{(5+d)/2}\exp\left(\frac{\lambda^{\varepsilon}A}{g}\right)\prod_{i=1}^{4}\left[\left(\frac{q_i^2}{\lambda^2}\right)\tilde{\phi}\left(\frac{q_i}{\lambda}\right)\right][1+{\rm O}(g,\varepsilon)], \tag{2.1}$$

where

$$A=\tfrac{8}{3}\pi^2+{\rm O}(\varepsilon)\,, \tag{2.2}$$

$$C_{\rm b}=2^{-1/2}\pi^{-3}\exp(3\varepsilon^{-1}+3\zeta'(2)\pi^{-2}-\tfrac{7}{2}\gamma-\tfrac{15}{4}), \tag{2.3}$$

$$\tilde{\phi}(q)=2^{d/2}\pi^{(d-2)/2}3^{1/2}|q|^{(2-d)/2}K_{(d-2)/2}(|q|). \tag{2.4}$$

Here $d=4-\varepsilon$ is the dimension of space, $\gamma$ is Euler's constant, $\zeta'(2)$ is the derivative of the Riemann zeta-function with argument 2 and $K$ is a modified Bessel function.

A number of comments need to be made in relation to (2.1). Firstly, although it is similar to the schematic expression (1.1), it differs in having an overall integration over the parameter $\lambda$. This is a remnant of the introduction of collective coordinates in the instanton calculation and represents an integration over the contribution of instantons of all scale sizes. Below we show that when the $\lambda$ integration is performed, a result of the form (1.1) is obtained for $\operatorname{Im}\Gamma^{(4)}$. Secondly, Drummond and Shore (1979) do not give explicit expressions for the imaginary part of vertex functions in their paper (only the partition function); however, it is straightforward to check that their slightly different approach to MW gives (2.1) but with $\tilde{\phi}(q)$ replaced by its value at $d=4$. Thirdly, we have put a subscript b on $\operatorname{Im}\Gamma^{(4)}$ to stress that this is the bare expression.

Before performing the $\lambda$ integration let us state that the integral converges for small $\lambda$ due to the exponential decay of the modified Bessel function $K$, but diverges logarithmically for large $\lambda$ as $\varepsilon$ goes to zero. This divergence is regulated by the factor in the exponential for $\varepsilon>0$, and manifests itself by a pole in $\varepsilon$ for small $\varepsilon$. We will see later that this pole may be interpreted as a non-perturbative ultraviolet divergence and can be removed by a non-perturbative renormalisation. The pole term is all that will interest us here, since we will use an extended minimal subtraction scheme.

To extract the pole term from (2.1) let us introduce the usual arbitrary, non-exceptional momentum scale $\mu$ as a lower cut-off and then expand $\tilde{\phi}(q_i/\lambda)$ for large $\lambda$:

$$\begin{aligned}\frac{q_i^2}{\lambda^2}\tilde{\phi}\left(\frac{q_i}{\lambda}\right)=2^2 3^{1/2}\pi\Bigg\{&\left(\frac{q_i}{2\lambda}\right)^{\varepsilon}[1+\tfrac{1}{2}\varepsilon(\gamma-\ln\pi)+{\rm O}(\varepsilon^2)]\\&+\left(\frac{q_i}{2\lambda}\right)^{2}\left[\left(\frac{q_i}{2\lambda}\right)^{\varepsilon}\left(\frac{2}{\varepsilon}+\gamma-\ln\pi+{\rm O}(\varepsilon)\right)-\left(\frac{2}{\varepsilon}+1-\gamma-\ln\pi+{\rm O}(\varepsilon)\right)\right]\\&+{\rm O}\left(\frac{q_i}{2\lambda}\right)^{4}\Bigg\}.\end{aligned} \tag{2.5}$$

This gives

$$-C_b\int_\mu^\infty \frac{d\lambda}{\lambda}\lambda^\varepsilon\left(-\frac{\lambda^\varepsilon A}{g}\right)^{(5+d)/2}\exp\left(\frac{\lambda^\varepsilon A}{g}\right)(2^2 3^{1/2}\pi)^4$$

$$\times\prod_{i=1}^4\left(\frac{q_i}{2\lambda}\right)^\varepsilon\left[1+O\left(\frac{q_i}{2\lambda}\right)^2\right][1+O(g,\varepsilon)]. \tag{2.6}$$

Only the leading term has been made explicit since the $O(1/\lambda^2)$ corrections give finite contributions at $\varepsilon=0$. The integral in (2.6) can be evaluated perturbatively in $g$:

$$\int_\mu^\infty \frac{d\lambda}{\lambda}\lambda^{a\varepsilon}\lambda^\varepsilon\exp\left[-\left(\frac{\lambda^\varepsilon A}{|g|}\right)\right]=\frac{1}{\varepsilon}\frac{|g|}{A}\mu^{a\varepsilon}\exp\left[-\left(\frac{\mu^\varepsilon A}{|g|}\right)\right][1+O(g)] \tag{2.7}$$

so that expression (2.6) can be written as

$$-2^8 3^2\pi^4 C_b\varepsilon^{-1}\left(-\frac{g}{A}\right)\left(-\frac{A\mu^\varepsilon}{g}\right)^{(5+d)/2}\exp\left(\frac{A\mu^\varepsilon}{g}\right)\prod_{i=1}^4\left(\frac{q_i}{\mu}\right)^\varepsilon[1+O(g,\varepsilon)]. \tag{2.8}$$

The factor $(q_i/\mu)^\varepsilon$ can be expanded in $\varepsilon$ and gives a typical finite contribution $\ln(q_i/\mu)$. If Drummond and Shore's instanton configuration is used, these momentum factors are not present; the expansion in $\varepsilon$ has already been made when expression (2.5) is replaced by the $\varepsilon\to 0$ limit in that approach. We are then left with

$$\operatorname*{Im}_{(\arg g=\pi)}\Gamma_b^{(4)}(q_i)=2^8 3^2\pi^4 C_b\varepsilon^{-1}\left(\frac{g}{A}\right)\left(-\frac{A\mu^\varepsilon}{g}\right)^{(5+d)/2}\exp\left(\frac{A\mu^\varepsilon}{g}\right)[1+O(g,\varepsilon)]. \tag{2.9}$$

This is now of the form (1.1) and we can begin renormalisation.

The real, perturbative part of $\Gamma^{(4)}(q_i)$ is easily evaluated to one loop. With our normalisation conventions (the interaction is $(g/4)\phi^4$ not $(g/4!)\phi^4$) one finds (see e.g. Amit 1978)

$$\operatorname{Re}\Gamma_b^{(4)}(q_i)=-6g+[27(4\pi^2\varepsilon)^{-1}+O(1)]g^2\mu^{-\varepsilon}+O(g^3), \tag{2.10}$$

where $\mu$ is the arbitrary momentum scale introduced in (2.6). Since the usual wavefunction renormalisation has no one-loop contribution ($Z^\phi=1+O(g^2)$) the expression (2.10) is rendered finite by a coupling constant renormalisation only:

$$g_r(\mu)=g\mu^{-\varepsilon}-9(8\pi^2\varepsilon)^{-1}g^2\mu^{-2\varepsilon}+O(g^3), \tag{2.11}$$

where we have used minimal subtraction and where 'r' stands for renormalised in the conventional perturbative sense. Using (2.9)–(2.11) we therefore have

$$\underset{(\arg g=\pi)}{\Gamma_r^{(4)}(q_i)}=-6\mu^\varepsilon\{g_r(\mu)[1+O(g_r)]+i2^7 3\pi^4 C_r\varepsilon^{-1}[-A/g_r(\mu)]^{(3+d)/2}$$

$$\times\exp[A/g_r(\mu)][1+O(g_r,\varepsilon)]\} \tag{2.12}$$

where

$$C_r=C_b\exp(-9A(8\pi^2\varepsilon)^{-1}). \tag{2.13}$$

Notice that $C_r$ is finite as $\varepsilon\to 0$ from (2.2) and (2.3); the only divergence that remains is the $1/\varepsilon$ pole in $\operatorname{Im}\Gamma_r^{(4)}$. It is now natural to perform a second renormalisation to eliminate this pole. We will show later in this section that a wavefunction renormalisation is also required but that it is of higher order in $g_r(\mu)$, and so as in (2.10) the pole in (2.12) is removed by a coupling constant renormalisation only:

$$g_{\mathrm{R}}(\mu)=g_{\mathrm{r}}(\mu)+\mathrm{i}2^7 3\pi^4 C_{\mathrm{r}}\varepsilon^{-1}[-A/g_{\mathrm{r}}(\mu)]^{(3+d)/2}\exp[A/g_{\mathrm{r}}(\mu)][1+\mathrm{O}(g_{\mathrm{r}})] \tag{2.14}$$

where R stands for the fully renormalised coupling constant. We have used here an extended minimal subtraction scheme in the sense that although $g_{\mathrm{R}}$ contains only the $1/\varepsilon$ divergent part, $A$ still depends upon $\varepsilon$, as given in (2.2). We shall see that the $\mathrm{O}(\varepsilon)$ correction to $A$ does not contribute to the results for exponents at this order. Therefore the undesirable expansion of $A$ as a power series in $\varepsilon$ within the exponential, which would give rise to $\mathrm{O}(\varepsilon/g)$ terms, is avoided. In the context of high-order estimates in the perturbative series, we interpret the extra pole in $\varepsilon$ as the one produced by the totally irreducible diagrams at high orders. These diagrams are known to be the dominant ones at $K$th order for $K$ large for $d<4$ and moreover diverge only like $1/\varepsilon$.

Having illustrated the general idea on the four-point function, we will now show how a similar two-step renormalisation can be applied to obtain the full renormalisation constants $Z^{\phi}$ and $Z^{\phi^2}$. However, there are new features in the calculations of these renormalisation constants. First, $\mathrm{Im}\,\Gamma^{(2)}$ is quadratically divergent. Second, there is a potentially disastrous momentum-dependent divergence in both the appropriate vertex functions. In perturbation theory, such a feature would appear only in two-loop or higher-order diagrams which had subdivergences; they are cancelled during the renormalisation process. A similar cancellation occurs in the non-perturbative terms here. Furthermore the remaining finite contributions are *not* the leading contributions to the imaginary parts of exponents. This indicates the internal consistency of these calculations; the systematics of the cancellation remain mysterious to us. The consequences are that the remainder of this section does not influence the results for exponents at this order.

Considering first $Z^{\phi}$, the analogous formula to (2.1) for $\mathrm{Im}\,\Gamma^{(2)}(q)$ is according to MW

$$\underset{(\arg g=\pi)}{\mathrm{Im}\,\Gamma_{\mathrm{b}}^{(2)}(q)}=C_{\mathrm{b}}\int_0^{\infty}\frac{\mathrm{d}\lambda}{\lambda}\lambda^2\left(-\frac{\lambda^{\varepsilon}A}{g}\right)^{(3+d)/2}\exp\left(\frac{\lambda^{\varepsilon}A}{g}\right)\left[\frac{q^2}{\lambda^2}\tilde{\phi}\left(\frac{q}{\lambda}\right)\right]^2[1+\mathrm{O}(g,\varepsilon)]. \tag{2.15}$$

The $\lambda^2$ factor in the integrand produces a quadratic divergence for $\varepsilon\to 0$. If the instanton configuration $\tilde{\phi}$ of Drummond and Shore (1979) is used, the quadratic divergence is momentum independent and can be removed by an appropriate mass counterterm. If the instanton configuration (2.5) of MW is used, then the same simple renormalisation is achieved only after (2.5) is expanded in $\varepsilon$ and higher-order terms discarded. (This is a loose end of this calculation which we have not been able to tidy up.) The subsequent consistency of the calculation to remove logarithmic divergences is some reassurance that this is the correct procedure. Following this procedure, we find that the required logarithmic divergence in $\partial\Gamma^{(2)}/\partial q^2$ is contained in the expression

$$C_{\mathrm{b}}\int_{\mu}^{\infty}\frac{\mathrm{d}\lambda}{\lambda}\left(-\frac{\lambda^{\varepsilon}A}{g}\right)^{(3+d)/2}\exp\left(\frac{\lambda^{\varepsilon}A}{g}\right)2^4 3\pi^2\left[\gamma-\tfrac{1}{2}+\ln\left(\frac{q}{2\lambda}\right)\right]\left[1+\mathrm{O}\left(\frac{q}{\lambda}\right)^2\right][1+\mathrm{O}(g,\varepsilon)] \tag{2.16}$$

in analogy to (2.6). Extracting the $1/\varepsilon$ pole by performing the leading $\lambda$ integral in (2.16), one obtains

$$\frac{\partial}{\partial q^2}\mathrm{Im}\,\Gamma_{\mathrm{b}}^{(2)}(q)=2^4\pi^2 3C_{\mathrm{b}}\varepsilon^{-1}\left[\gamma-\tfrac{1}{2}+\ln\left(\frac{q}{2\mu}\right)\right]\left(-\frac{A\mu^{\varepsilon}}{g}\right)^{(1+d)/2}\times\exp[A\mu^{\varepsilon}/g][1+\mathrm{O}(g,\varepsilon)]. \tag{2.17}$$

This is very similar to the expression (2.9) for Im $\Gamma_b^{(4)}(q_i)$; however, notice the appearance of a $\varepsilon^{-1}\ln(q/\mu)$ term in (2.17). We shall show below how this term gets cancelled by a contribution from the perturbative part of $\Gamma_b^{(2)}(q)$, during the second stage of renormalisation.

In order to illustrate the cancellation of this term it is necessary to use the two-loop expression for the real perturbative part of $\Gamma^{(2)}(q)$:

$$\mathrm{Re}\,\Gamma_b^{(2)}(q)=q^2+q^{2-2\varepsilon}g^2[3/4(8\pi^2)^2\varepsilon+\mathrm{O}(1)]+\mathrm{O}(g^3) \tag{2.18}$$

or

$$(\partial/\partial q^2)\,\mathrm{Re}\,\Gamma_b^{(2)}(q)=1+q^{-2\varepsilon}g^2[3/4(8\pi^2)^2\varepsilon+P+\mathrm{O}(\varepsilon)]+\mathrm{O}(g^3) \tag{2.19}$$

where, for later convenience, the order-one term $P$ has been made explicit. The perturbative wavefunction renormalisation may be easily read off:

$$Z_p^{\phi}=1-\frac{3}{4(8\pi^2)^2\varepsilon}g^2\mu^{-2\varepsilon}+\mathrm{O}(g^3)=1-\frac{3}{4(8\pi^2)^2\varepsilon}g_r^2(\mu)+\mathrm{O}(g_r^3) \tag{2.20}$$

and so after the first stage of renormalisation we obtain

$$\begin{aligned}\frac{\partial}{\partial q^2}\Gamma^{(2)}_{(\arg g=\pi)}(q)&=1+g_r^2((\mu)\left[P-\frac{3}{2(8\pi^2)^2}\ln\left(\frac{q}{\mu}\right)+\mathrm{O}(g_r^2)\right.\\&\quad+\mathrm{i}2^4 3\pi^2C_r\varepsilon^{-1}\left[\gamma-\tfrac{1}{2}-\ln 2+\ln\left(\frac{q}{\mu}\right)\right]\left(-\frac{A}{g_r(\mu)}\right)^{(1+d)/2}\\&\quad\times\exp\left(\frac{A}{g_r(\mu)}\right)[1+\mathrm{O}(g_r,\varepsilon)].\end{aligned} \tag{2.21}$$

Writing this expression in terms of the fully renormalised coupling constant (2.14) gives

$$\begin{aligned}\frac{\partial}{\partial q^2}\Gamma^{(2)}_{r\,(\arg g=\pi)}(q)&=1+g_R^2(\mu)\left[P-\frac{3}{2(8\pi^2)^2}\ln\left(\frac{q}{\mu}\right)+\mathrm{O}(\varepsilon)\right]+\mathrm{O}(g_R^3)\\&\quad+\mathrm{i}2^4 3\pi^2C_r\varepsilon^{-1}\left[2^4\pi^2A\left(P-\frac{3}{2(8\pi^2)^2}\ln\frac{q}{\mu}\right)+\gamma-\tfrac{1}{2}-\ln 2+\ln\left(\frac{q}{\mu}\right)+\mathrm{O}(\varepsilon)\right]\\&\quad\times\left(-\frac{A}{g_R(\mu)}\right)^{(1+d)/2}\exp\left(\frac{A}{g_R(\mu)}\right)[1+\mathrm{O}(g_R)].\end{aligned} \tag{2.22}$$

The $\varepsilon^{-1}\ln(q/\mu)$ cancel as promised and therefore

$$\begin{aligned}\frac{\partial}{\partial q^2}\Gamma^{(2)}_{r\,(\arg g=\pi)}(q)&=1+g_R^2(\mu)\left[P-\frac{3}{2(8\pi^2)^2}\ln\left(\frac{q}{\mu}\right)+\mathrm{O}(\varepsilon)\right]+\mathrm{O}(g_R^3)\\&\quad+\frac{iC_r}{\varepsilon}2^4\pi^2 3(2^4\pi^2AP+\gamma-\tfrac{1}{2}-\ln 2)\left(-\frac{A}{g_R(\mu)}\right)^{(1+d)/2}\\&\quad\times\exp\left(\frac{A}{g_R(\mu)}\right)[1+\mathrm{O}(g_R,\varepsilon)].\end{aligned} \tag{2.23}$$

One can now define a non-perturbative minimally subtracted wavefunction renormalisation to cancel the $1/\varepsilon$ pole:

$$-Z^{\phi}_{\rm np}=1+\frac{{\rm i}C_{\rm r}}{\varepsilon}2^4\pi^2 3(2^4\pi^2 AP+\gamma-\tfrac{1}{2}-\ln 2)\left(-\frac{A}{g_{\rm R}(\mu)}\right)^{(1+d)/2}$$
$$\times\exp\left(\frac{A}{g_{\rm R}(\mu)}\right)[1+{\rm O}(g_{\rm R})]. \tag{2.24}$$

The full wavefunction renormalisation, $Z^{\phi}=Z^{\phi}_{\rm p}Z^{\phi}_{\rm np}$, may be found from (2.20) and (2.24).

The renormalisation of the vertex function with a $\frac{1}{2}\phi^2$ insertion, $\Gamma^{(2,1)}(q_1, q_2; q_3)$, remains. This particular function was not discussed in MW but a straightforward application of the techniques discussed there yields

$$\underset{(\arg g=\pi)}{\operatorname{Im}\Gamma^{(2,1)}_{\rm b}(q_1,q_2;q_3)}$$
$$=-\tfrac{1}{2}C_{\rm b}\int_0^\infty\frac{{\rm d}\lambda}{\lambda}\left(-\frac{\lambda^\varepsilon A}{g}\right)^{(5+d)/2}\exp\left(\frac{\lambda^\varepsilon A}{g}\right)\prod_{i=1}^{2}\left[\frac{q_i^2}{\lambda^2}\tilde{\phi}\left(\frac{q_i}{\lambda}\right)\right]$$
$$\times\tilde{\tilde{\phi}}\left(\frac{q_3}{\lambda}\right)[1+{\rm O}(g,\varepsilon)], \tag{2.25}$$

where

$$\tilde{\tilde{\phi}}(q)=\frac{3}{\pi^2}\int {\rm d}^d x\frac{{\rm e}^{{\rm i}qx}}{(1+x^2)^2}=2^{(d-2)/2}\pi^{(d-4)/2}3|q|^{(4-d)/2}K_{(d-4)/2}(|q|) \tag{2.26}$$

and all other quantities are as before. The required divergence is contained in

$$-\tfrac{1}{2}C_{\rm b}\int_\mu^\infty\frac{{\rm d}\lambda}{\lambda}\left(-\frac{\lambda^\varepsilon A}{g}\right)^{(5+d)/2}\exp\left(\frac{\lambda^\varepsilon A}{g}\right)2^5\pi^2 3^2\left[-\gamma-\ln\left(\frac{q_3}{2\lambda}\right)\right][1+{\rm O}(g,\varepsilon)]$$
$$=C_{\rm b}2^4\pi^2 3^2\left(-\frac{A\mu^\varepsilon}{g}\right)^{(3+d)/2}\left[\gamma+\ln\left(\frac{q_3}{2\mu}\right)\right]\frac{1}{\varepsilon}$$
$$\times\exp\left(\frac{A\mu^\varepsilon}{g}\right)[1+{\rm O}(g,\varepsilon)]. \tag{2.27}$$

Including the real perturbative part of $\Gamma^{(2,1)}_{\rm b}(q_1, q_2; q_3)$ we therefore have

$$\underset{(\arg g=\pi)}{\Gamma^{(2,1)}_{\rm b}(q_1,q_2;q_3)}=1+gq_3^{-\varepsilon}\left(-\frac{3}{8\pi^2\varepsilon}+Q+{\rm O}(\varepsilon)\right)+{\rm O}(g^2)$$
$$+{\rm i}C_{\rm b}2^4\pi^2 3^2\frac{1}{\varepsilon}\left[\gamma+\ln\left(\frac{q_3}{2\mu}\right)\right]\left(-\frac{A\mu^\varepsilon}{g}\right)^{(3+d)/2}$$
$$\times\exp\left(\frac{A\mu^\varepsilon}{g}\right)[1+{\rm O}(g,\varepsilon)] \tag{2.28}$$

where $Q$ is some number which will not be required. We now proceed exactly as we did in the case of wavefunction renormalisation. The perturbative renormalisation of the $\phi^2$ operator ($\Gamma^{(2,1)}_{\rm r}=Z^{\phi}Z^{\phi^2}\Gamma^{(2,1)}_{\rm b}$, as in Amit (1978)) is

$$Z^{\phi^2}_{\rm p}=1+3g\mu^{-\varepsilon}/(8\pi^2\varepsilon)+{\rm O}(g^2)=1+3g_{\rm r}(\mu)/(8\pi^2\varepsilon)+{\rm O}(g_{\rm r}^2) \tag{2.29}$$

and after this first stage of renormalisation we obtain

$$\Gamma_{\mathrm{r}\,(\arg g=\pi)}^{(2,1)}(q_1, q_2; q_3) = 1 + g_{\mathrm{r}}(\mu)\left[Q + \frac{3}{8\pi^2}\ln\left(\frac{q_3}{\mu}\right) + \mathrm{O}(\varepsilon)\right] + \mathrm{O}(g_{\mathrm{r}}^2)$$
$$+ \mathrm{i}C_{\mathrm{r}}\frac{2^4\pi^2 3^2}{\varepsilon}\left[\gamma + \ln\left(\frac{q_3}{2\mu}\right)\right]\left(-\frac{A}{g_{\mathrm{r}}(\mu)}\right)^{(d+3)/2}$$
$$\times \exp\left(\frac{A}{g_{\mathrm{r}}(\mu)}\right)[1 + \mathrm{O}(g_{\mathrm{r}}, \varepsilon)]. \tag{2.30}$$

Introducing the fully renormalised coupling constant $g_{\mathrm{R}}(\mu)$, the logs cancel, and we are left with

$$\Gamma_{\mathrm{r}\,(\arg g=\pi)}^{(2,1)}(q_1, q_2; q_3) = 1 + g_{\mathrm{R}}(\mu)\left[Q + \frac{3}{8\pi^2}\ln\left(\frac{q_3}{\mu}\right) + \mathrm{O}(\varepsilon)\right] + \mathrm{O}(g_{\mathrm{R}}^2)$$
$$- \frac{\mathrm{i}C_{\mathrm{r}}}{\varepsilon}2^4\pi^2 3^2(\tfrac{8}{3}\pi^2 Q - \gamma + \ln 2)\left(-\frac{A}{g_{\mathrm{R}}(\mu)}\right)^{(d+3)/2}$$
$$\times \exp\left(\frac{A}{g_{\mathrm{R}}(\mu)}\right)[1 + \mathrm{O}(g_{\mathrm{R}}, \varepsilon)]. \tag{2.31}$$

The final $1/\varepsilon$ pole is now removed by defining a non-perturbative $Z^{\phi^2}$ as

$$Z_{\mathrm{np}}^{\phi} Z_{\mathrm{np}}^{\phi^2} = 1 + \frac{\mathrm{i}C_{\mathrm{r}}}{\varepsilon}2^4\pi^2 3(\tfrac{8}{3}\pi^2 Q - \gamma + \ln 2)\left(-\frac{A}{g_{\mathrm{R}}(\mu)}\right)^{(d+3)/2}$$
$$\times \exp\left(\frac{A}{g_{\mathrm{R}}(\mu)}\right)[1 + \mathrm{O}(g_{\mathrm{R}})]. \tag{2.32}$$

This is equal to $Z_{\mathrm{np}}^{\phi^2}$ at this order, as is seen from (2.24). The full renormalisation constant $Z^{\phi^2}$ is the product of this factor and the one given by (2.29).

## 3. Renormalisation group functions, fixed points and critical exponents

In § 2 renormalisation constants were obtained which had both real and imaginary parts. The real parts were just the usual renormalisation constants calculated in perturbation theory, whereas the imaginary parts, which only existed for $g < 0$, corresponded to an additional renormalisation required to remove a new divergence obtained from the integration over instanton scale sizes. Using these constants we will now calculate the renormalisation group functions $\beta(g_{\mathrm{R}})$, $\gamma(g_{\mathrm{R}})$ and $\gamma_{\phi^2}(g_{\mathrm{R}})$ in the usual way. These functions will be finite as $\varepsilon \to 0$, but will have an imaginary part for negative coupling.

We begin with the $\beta$ function,

$$\beta(g_{\mathrm{R}}) = \mu(\partial/\partial\mu)g_{\mathrm{R}}(\mu)$$
$$= \mu\frac{\partial}{\partial\mu}g_{\mathrm{r}}(\mu) + \frac{\mathrm{i}C_{\mathrm{r}}}{\varepsilon}2^7 3\pi^4\mu\frac{\partial}{\partial\mu}\left[\left(-\frac{A}{g_{\mathrm{r}}(\mu)}\right)^{(3+d)/2}\right.$$
$$\left.\times \exp\left(\frac{A}{g_{\mathrm{r}}(\mu)}\right)[1 + \mathrm{O}(g_{\mathrm{r}})]\right] \tag{3.1}$$

where the expression (2.14) for $g_R(\mu)$ has been used and the derivative is to be taken at fixed bare coupling $g$. Therefore

$$\beta(g_R)=\beta_p(g_r)-iC_r2^73\pi^4\left(-\frac{A}{g_r(\mu)}\right)^{(5+d)/2}\exp\left(\frac{A}{g_r(\mu)}\right)[1+O(g_r)]. \tag{3.2}$$

This expression can also be obtained directly from (2.6), the $\mu\partial/\partial\mu$ derivative eliminating the integration. The real part of $\beta(g_R)$ is $\beta_p(g_r)$, the familiar perturbative $\beta$-function:

$$\beta_p(g_r)=-\varepsilon g_r+(9/8\pi^2)g_r^2+O(g_r^3). \tag{3.3}$$

It is easy to check, using (2.14), that to this order $g_r$ may be replaced by $g_R$ in (3.2), and thus

$$\beta(g_R)=-\varepsilon g_R+\frac{9}{8\pi^2}g_R^2+O(g_R^3)-iC_r2^7\pi^43\left(-\frac{A}{g_R}\right)^{(5+d)/2}\exp\left(\frac{A}{g_R}\right)[1+O(g_R)]. \tag{3.4}$$

The anomalous dimension of the field, $\gamma(g_R)$, is defined by

$$\gamma(g_R)=\mu(\partial/\partial\mu)\ln Z^\phi=\mu(\partial/\partial\mu)(\ln Z_p^\phi+\ln Z_{np}^\phi). \tag{3.5}$$

The results (2.20) and (2.24) for $Z_p^\phi$ and $Z_{np}^\phi$, respectively, lead to

$$\begin{aligned}\gamma(g_R)&=\frac{3}{2(8\pi^2)^2}g_r^2+O(g_r^3)+iC_r2^4\pi^23(2^7\pi^43^{-1}P+\gamma-\tfrac{1}{2}-\ln 2)\\&\quad\times\left(-\frac{A}{g_R}\right)^{(d+3)/2}\exp\left(\frac{A}{g_R}\right)[1+O(g_R)]\\&=\frac{3}{2(8\pi^2)^2}g_R^2+O(g_R^3)+iC_r2^4\pi^23(2^7\pi^43^{-1}P+\gamma-\tfrac{1}{2}-\ln 2)\\&\quad\times\left(-\frac{A}{g_R}\right)^{(d+3)/2}\exp\left(\frac{A}{g_R}\right)[1+O(g_R)]\end{aligned} \tag{3.6}$$

to this order. Similarly

$$\gamma_{\phi^2}(g_R)=-\mu(\partial/\partial\mu)\ln Z^{\phi^2}=-\mu(\partial/\partial\mu)(\ln Z_p^{\phi^2}+\ln Z_{np}^{\phi^2}) \tag{3.7}$$

is found from (2.29) and (2.32) to be

$$\frac{3g_r}{8\pi^2}+O(g_r^2)+iC_r2^4\pi^23^2(\tfrac{8}{3}\pi^2Q-\gamma+\ln 2)\left(-\frac{A}{g_R}\right)^{(d+5)/2}\exp\left(\frac{A}{g_R}\right)[1+O(g_R)]. \tag{3.8}$$

Thus to this order

$$\begin{aligned}\gamma_{\phi^2}(g_r)&=\frac{3g_R}{8\pi^2}+O(g_R^2)+iC_r2^4\pi^23^2(\tfrac{8}{3}\pi^2Q-\gamma+\ln 2)\left(-\frac{A}{g_R}\right)^{(d+5)/2}\\&\quad\times\exp\left(\frac{A}{g_R}\right)[1+O(g_R)].\end{aligned} \tag{3.9}$$

Note that these renormalisation group functions have the form we expect within the extended minimal subtraction scheme: $\gamma$ and $\gamma_{\phi^2}$ are independent of $\varepsilon$ and $\beta(g_R)=-\varepsilon g_R+\beta_4(g_R)$, $\beta_4$ being the $\beta$ function in four dimensions, with the understanding that there is a hidden $\varepsilon$ dependence in $A$.

It is the zeros of the function (3.4) that are of special interest to us. Denoting these by $g_R^*$ we have

$$0=-\varepsilon+\frac{9}{8\pi^2}g_R^*+O(g_R^{*2})+iC_r2^43^2\pi^2\left(-\frac{A}{g_R^*}\right)^{(7+d)/2}\exp\left(\frac{A}{g_R^*}\right)[1+O(g_R^*)] \tag{3.10}$$

if $g_R^*\neq 0$. Writing $g_R^*=g_1^*+ig_2^*$, $g_1^*$ and $g_2^*$ both real, gives to this order

$$0=-\varepsilon+(9/8\pi^2)g_1^*, \tag{3.11}$$

$$0=(9/8\pi^2)g_2^*+C_r2^43^2\pi^2(-A/g_1^*)^{(7+d)/2}\exp(A/g_1^*). \tag{3.12}$$

But $g_1^*=g_r^*$ is negative and thus only if $\varepsilon<0$ does a non-trivial fixed point exist. In this case

$$g_2^*=-C_r2^7\pi^4(-A/g_1^*)^{(7+d)/2}\exp(A/g_1^*)[1+O(\varepsilon)]. \tag{3.13}$$

To obtain the correct overall constant in (3.13) when we substitute for $g_1^*$, we have to use the two-loop expression for $g_1^*$ (in minimal subtraction, to be consistent with (2.11) *et seq*):

$$g_1^*=\tfrac{8}{9}\pi^2\varepsilon[1+\tfrac{17}{27}\varepsilon+O(\varepsilon^2)] \tag{3.14}$$

so that

$$\begin{aligned}g_2^*&=-C_r2^7\pi^4(-3/\varepsilon)^{(7+d)/2}\exp[(9A/8\pi^2\varepsilon)(1-\tfrac{17}{27}\varepsilon)][1+O(\varepsilon)]\\&=C_g(-3/\varepsilon)^{(7+d)/2}\exp(3/\varepsilon)[1+O(\varepsilon)]\end{aligned} \tag{3.15}$$

where

$$C_g=-2^{13/2}\pi\exp(3\zeta'(2)\pi^{-2}-\tfrac{7}{2}\gamma-\tfrac{203}{36}). \tag{3.16}$$

Note that the explicit $A$- dependent factor in $C_r$ in (2.13) cancels a corresponding factor from $\exp(A/g_1^*)$ and only the O(1) part of $A$ (equation (2.2)) is required as stated earlier. We remark also that $C_g$ is independent of the perturbative renormalisation scheme chosen. (The argument is the same as that presented in Bruce and Wallace (1983, p 1732 *et seq*).)

Thus we have found that, assuming $\beta(g_R)=-\varepsilon g_R+\beta_4(g_R)$ holds for all $\varepsilon$ (in particular for $\varepsilon<0$), there is a non-trivial fixed point of the form

$$\underset{(\arg\varepsilon=\pi)}{g_R^*}=\tfrac{8}{9}\pi^2\varepsilon+O(\varepsilon^2)+iC_g(-3/\varepsilon)^{(7+d)/2}\exp(3/\varepsilon)[1+O(\varepsilon)]. \tag{3.17}$$

When this value of $g_R$ is substituted into $\gamma(g_R)$ and $\gamma_{\phi^2}(g_R)$ two terms which are non-perturbative in $\varepsilon$ appear. In both cases the one which comes from substituting $g_1^*$ into the imaginary part of these anomalous dimensions is down by a factor of $\varepsilon$ on the one which comes from substituting $g_2^*$ into the real part of the anomalous dimension. Thus as remarked earlier, the results from the latter part of § 2 do not enter and so to this order

$$\eta\equiv\gamma(g_R^*)=3\pi^{-4}2^{-7}g_R^{*2}, \tag{3.18}$$

$$-[\nu^{-1}-2]\equiv\gamma_{\phi^2}(g_R^*)=3\pi^{-2}2^{-3}g_R^*. \tag{3.19}$$

Using the explicit form (3.17), the above two equations imply that

$$\underset{(\arg\varepsilon=\pi)}{\operatorname{Im}\eta(\varepsilon)}=C_\eta(-3/\varepsilon)^{(5+d)/2}\exp(3/\varepsilon)[1+O(\varepsilon)], \tag{3.20}$$

$$\operatorname*{Im}_{(\arg\varepsilon=\pi)}[\nu^{-1}(\varepsilon)]=C_{\nu^{-1}}(-3/\varepsilon)^{(7+d)/2}\exp(3/\varepsilon)[1+\mathrm{O}(\varepsilon)], \tag{3.21}$$

where

$$C_{\nu^{-1}}=3C_\eta=2^{7/2}3\pi^{-1}\exp[3\zeta'(2)\pi^{-2}-\tfrac{7}{2}\gamma-\tfrac{203}{36}]. \tag{3.22}$$

Finally, defining the exponent $\omega=\beta'(g_{\mathrm{R}})|_{g_{\mathrm{R}}=g_{\mathrm{R}}^*}$, it is straightforward to find that

$$\operatorname*{Im}_{(\arg\varepsilon=\pi)}\omega(\varepsilon)=C_\omega\left(-\frac{3}{\varepsilon}\right)^{(9+d)/2}\exp\left(\frac{3}{\varepsilon}\right)[1+\mathrm{O}(\varepsilon)]\qquad :C_\omega=9C_\eta. \tag{3.23}$$

The functions $\eta(\varepsilon)$, $\nu^{-1}(\varepsilon)$ and $\omega(\varepsilon)$ are the conventional critical exponents when $\varepsilon>0$ $(d<4)$. In the above, the functions are defined in the same way, but for negative values of $\varepsilon$ $(d>4)$. The instability of the theory due to the negative coupling is transferred, at the fixed point, to an instability for negative $\varepsilon$ which manifests itself in the generation of imaginary parts for the critical exponents. These imaginary parts are exponentially small in $\varepsilon$, in the usual way. Moreover they are expected to be universal functions, in the sense that they are independent of the renormalisation scheme, and the above calculation confirms this to lower order. In § 4 we will write a dispersion relation in $\varepsilon$ which will relate the imaginary parts of these critical exponents for $\varepsilon<0$ to their high-order behaviour for the physically interesting (i.e. $\varepsilon>0$) sector of the theory.

Before going on to this, however, we wish to outline a slightly different way of obtaining the functions $\eta(\varepsilon)$ and $\nu^{-1}(\varepsilon)$ for $\varepsilon<0$, which leads to the same results.

The spirit of this alternative approach is the same as that used in the Feynman graph method for perturbative calculation of critical exponents (Wilson 1972). In this method one works with the bare theory, with a cut-off $\Lambda$ which makes the loop integrals finite in four dimensions. The integrands can then be expanded in $\varepsilon$, so that one obtains, perturbatively, a double expansion in the bare coupling $g$ and $\varepsilon$. The fixed point value of $g^*$ (for a renormalisation group discussion see Zinn-Justin (1973), Amit (1978, p 214)) can be determined as a power series in $\varepsilon$ by demanding that the sum of logarithms (e.g. $\ln(q/\Lambda)$) exponentiates to the expected scaling behaviour, in particular that the massless four-point vertex function scales as

$$\Gamma^{(4)}(q,g^*,\Lambda)\propto q^{\varepsilon-2\eta}\sim q^{\varepsilon}, \tag{3.24}$$

to lowest order.

We cannot pursue this approach completely here, because the determinant calculation resulting in (2.1) is done using dimensional regularisation without a cut-off. Therefore we must still perform the standard perturbative renormalisation in order to remove the divergences from perturbative fluctuations. Renormalising (2.1) with the coupling at momentum scale $\lambda$, one then obtains (MW, equation (42))

$$\operatorname{Im}\Gamma^{(4)}=-C_r\int_0^\infty\frac{\mathrm{d}\lambda}{\lambda}\lambda^{\varepsilon}\left(-\frac{A}{g_r(\lambda)}\right)^{(5+d)/2}\times\exp\left(\frac{A}{g_r(\lambda)}\right)\prod_{i=1}^{4}\left[\frac{q_i^2}{\lambda^2}\tilde{\phi}\left(\frac{q_i}{\lambda}\right)\right][1+\mathrm{O}(g,\varepsilon)]. \tag{3.25}$$

We can then ask if it is possible to find the non-perturbative (in $\varepsilon$) part of $g^*$ for which (3.24) holds.

Several new aspects then emerge. First, $g_r$ is to be set to the fixed point value in this approach, $g_r(\lambda)=g_r^*$ independent of $\lambda$. Therefore there is now no convergence factor from the exponential term in the integrand. Second, in the spirit of the Feynman graph approach, we expand the integrand (the final relic of the one-loop integral) in $\varepsilon$. Then from (2.4) we have

$$\operatorname{Im}\Gamma^{(4)}(q, g^*) = -C_r\left(-\frac{A}{g_1^*}\right)^{(5+d)/2} \exp\left(\frac{A}{g_1^*}\right)\int_0^\infty \frac{d\lambda}{\lambda} \times\prod_{i=1}^{4}\left[2^2\pi 3^{1/2}\left(\frac{|q_i|}{\lambda}\right)K_1\left(\frac{|q_i|}{\lambda}\right)\right][1+O(\varepsilon)]. \tag{3.26}$$

The integral is convergent as before for $\lambda \to 0$ but diverges logarithmically for $\lambda \to \infty$. Therefore we must impose an explicit cut-off $\Lambda$ on the upper limit of integration. Such a cut-off is a perfectly good ultraviolet regulator, respecting internal symmetries and breaking dilatation invariance as it must. One then obtains for, say, a symmetry point of momentum $q$

$$\operatorname{Im}\Gamma^{(4)}(q, g^*) = -C_r 2^8\pi^4 3^2\left(-\frac{A}{g_1^*}\right)^{(5+d)/2} \exp\left(\frac{A}{g_1^*}\right)\left(\ln\frac{\Lambda}{q}+\text{finite}\right)[1+O(\varepsilon)]. \tag{3.27}$$

The value of the imaginary part, $g_2^*$, of the fixed point coupling follows by combining the $\ln q$ from the $O(g^2)$ one-loop graph with the $\ln q$ in (3.27) to yield the expansion $1+\varepsilon\ln q$ of (3.24). The result agrees with that obtained in (3.13). Since only $\operatorname{Im} g_R^*$ is required to find $\operatorname{Im}\eta$ and $\operatorname{Im}\nu^{-1}$ (see (3.18) and (3.19)) these results are also recovered using this approach.

## 4. High-order behaviour of critical exponents in the $\varepsilon$ expansion

Critical exponents, when calculated using the field-theoretic renormalisation group and the $\varepsilon$ expansion, are just the values of the renormalisation group functions at fixed points (zeros of the $\beta$ function). For physical applications $\varepsilon$ is real and positive; however, more generally one may define the exponents for complex $\varepsilon$.

In § 3 we saw that if, in particular, we take $\varepsilon$ real and negative, then the exponents have an imaginary part whose sign depends on whether $\arg\varepsilon$ is $+\pi$ or $-\pi$. This analytic structure is inherited from the renormalisation group functions, which have a branch cut along the negative coupling constant axis. Therefore just as one is able to exploit this structure and write dispersion relations to obtain the high-order behaviour of functions expanded in terms of the coupling (see MW, for example), we will assume that the real part of a particular critical exponent $F(\varepsilon)$ is related to the imaginary part by

$$\operatorname{Re} F(\varepsilon) = \frac{1}{\pi}\int_{-\infty}^{0}\frac{\operatorname{Im} F(\varepsilon')}{\varepsilon'-\varepsilon}\,d\varepsilon', \qquad \varepsilon > 0. \tag{4.1}$$

If the perturbative part of $F(\varepsilon)$ is written as

$$F(\varepsilon) \sim \sum_{K=0}^{\infty} F_K\varepsilon^K, \qquad \varepsilon > 0, \tag{4.2}$$

then from (4.1) we have

$$F_K = \frac{1}{\pi}\int_{-\infty}^{0}\frac{\operatorname{Im} F(\varepsilon')\,d\varepsilon'}{(\varepsilon')^{K+1}}. \tag{4.3}$$

In § 3 the form of the imaginary part of the exponents $\eta$, $\nu^{-1}$ and $\omega$ was calculated and found to have the general form

$$\operatorname*{Im}_{(\arg \varepsilon'=\pi)} F(\varepsilon') = C(-1/a\varepsilon')^{\alpha-\varepsilon'/2} \exp(1/a\varepsilon')[1+O(\varepsilon')] \tag{4.4}$$

where $a$, $\alpha$ and $C$ were known constants. Moreover, in the appendix the calculation is extended to cover the $n$-component model and the same form is found; only the constants differ. Thus in what follows we will work for general $n$.

Substituting (4.4) into (4.3) and performing the integral for large $K$ by steepest descents we find

$$F_K = -(C/\pi)K!(-a)^K K^{\alpha-1}[1+O((\ln K)/K)]. \tag{4.5}$$

The $O(\varepsilon')$ terms in (4.4) give $O(1/K)$ corrections to (4.5); the $O((\ln K)/K)$ corrections come from the $(-1/a\varepsilon')^{-\varepsilon'/2}$ term. The structure of (4.5) is as in (1.2), the overall constant $C$ being the result of a one-loop calculation about the instanton.

The results from the appendix together with (4.5) give the $K$th-order term ($K$ large) in the $\varepsilon$ expansion for $\eta$, $\nu^{-1}$ and $\omega$ as

$$\eta_K = -\bar{C}_\eta K![-3/(n+8)]^K K^{(n+6)/2}[1+O((\ln K)/K)], \tag{4.6}$$

$$\nu_K^{-1} = -\bar{C}_{\nu^{-1}} K![-3/(n+8)]^K K^{(n+8)/2}[1+O((\ln K)/K)], \tag{4.7}$$

$$\omega_K = -\bar{C}_\omega K![-3/(n+8)]^K K^{(n+10)/2}[1+O((\ln K)/K)], \tag{4.8}$$

where

$$\bar{C}_{\nu^{-1}} = 3\bar{C}_\eta, \qquad \bar{C}_\omega = [3(n+8)/(n+2)]\bar{C}_\eta, \tag{4.9}$$

$$\bar{C}_\eta = \frac{2^{-(n-16)/6} 3^{(n+3)/2} \pi^{-(n+8)/6}}{(n+8)\Gamma(\frac{1}{2}n+1)} \times \exp\left(\frac{(n+2)\zeta'(2)}{\pi^2} - \tfrac{1}{2}(n+6)\gamma - \tfrac{1}{4}(n+14) - \frac{(3n+14)}{(n+8)}\right). \tag{4.10}$$

Values for $\bar{C}_\eta$, $\bar{C}_{\nu^{-1}}$ and $\bar{C}_\omega$ for $n = 0$, 1, 2 and 3 are shown in table 1, from which we see that $\bar{C}_\eta$ is of order $10^{-4}$ or $10^{-5}$.

**Table 1.** Approximate numerical values for the overall constants in the high-order estimates for the critical exponents $\eta$, $\nu^{-1}$ and $\omega$.

| $n$ | $\bar{C}_\eta$ | $\bar{C}_{\nu^{-1}}$ | $\bar{C}_\omega$ |
|---|---|---|---|
| 0 | $6.885\times10^{-4}$ | $2.066\times10^{-3}$ | $8.262\times10^{-3}$ |
| 1 | $4.067\times10^{-4}$ | $1.220\times10^{-3}$ | $3.660\times10^{-3}$ |
| 2 | $1.964\times10^{-4}$ | $5.892\times10^{-4}$ | $1.473\times10^{-3}$ |
| 3 | $8.299\times10^{-5}$ | $2.490\times10^{-4}$ | $5.477\times10^{-4}$ |

## 5. Conclusion

The results given in § 4 were only made possible by performing the renormalisation of the model with $\varepsilon \neq 0$. In MW, following Brézin *et al* (1977a), the second stage of renormalisation was carried out in exactly four dimensions. No extra divergence was found because it was tacitly assumed that $|g_r| > |g_r^*| = O(\varepsilon)$, a condition that is required if the limit $\varepsilon \to 0$ is to be safely taken. In this paper we have been interested in fixed point quantities with finite $\varepsilon$ and have consequently taken $|g_r| \leq |g_r^*| = O(\varepsilon)$. This condition is automatically satisfied if both $g_r$ and $\varepsilon$ have the same sign, and leads to different behaviour to that found if the former condition is used. Our method for dealing with the extra non-perturbative divergence is nevertheless no more than a prescription—no proof that our scheme extends to higher orders has been given. We should recall, however, that the imaginary part of the vertex function contains the same information that is contained in the real part, it is just ordered differently. The knowledge that the leading-order term in the imaginary part corresponds to the totally irreducible diagrams of high order gives us an understanding of why we should find a $1/\varepsilon$ divergence, even after a one-loop perturbative renormalisation. This, together with the cancellation of $\varepsilon^{-1}\ln(q/\mu)$ terms, the reproduction of the previously known structure for the renormalisation group functions and critical exponents and the fact that we obtain the same result with a cut-off directly in $d > 4$, gives us confidence that our method is correct.

There are other advantages in our approach. Firstly, no problems should arise because of extra singularities (called renormalons by 't Hooft (1977)) which exist in the vertex functions as a function of $g$, for $d = 4$. These are not present in our renormalisation group functions as they are calculated away from $d = 4$ (the fact that the minimally subtracted renormalisation group functions are the same—apart from the $-\varepsilon g_R$ factor in $\beta(g_R)$—for $\varepsilon \neq 0$ and $\varepsilon = 0$ is an indication that these functions calculated in minimal subtraction are free of renormalon singularities). Secondly, by writing a dispersion relation in $\varepsilon$, no decision had to be made as to whether one should disperse in $g$, $g_r$ or $g_R$, that is, how renormalisation and dispersion interacted. The imaginary parts of the critical exponents are universal functions of an unambiguous parameter $\varepsilon$ and as such should be safer to handle.

Finally, it is clear in principle what has to be done to extend the above work to higher orders. In practice the effort required may be prohibitive.

## Acknowledgments

We thank G Parisi and J Zinn-Justin for useful discussions. DJW acknowledges support and hospitality from the Departamento de Fisica, UFPE, Recife, where part of this work was done. AJM acknowledges financial support from the SERC.

## Appendix

In this appendix we will do little more than state the generalisations of the results of §§ 2 and 3 to the case of an $O(n)$ symmetric theory with interaction $\frac{1}{4}g(\phi^2)^2 = \frac{1}{4}g(\sum_{i=1}^n \phi_i\phi_i)^2$.

The form of the imaginary parts of the vertex functions in the general $n$ case are the same as those of the $n = 1$ model, for example (equation (A14) of MW)

$$\operatorname*{Im}_{(\arg g = \pi)} \Gamma^{(4)}_{\mathrm{b},ijkl}(q) = -C_{\mathrm{b}}^{(n)} \int \frac{\mathrm{d}\lambda}{\lambda} \lambda^{\varepsilon} \left(-\frac{\lambda^{\varepsilon} A}{g}\right)^{(d+n+4)/2} \exp\left(\frac{\lambda^{\varepsilon} A}{g}\right) \times \prod_{\alpha=1}^{4} \left[\frac{q_\alpha^2}{\lambda^2} \tilde{\phi}\left(\frac{q_\alpha}{\lambda}\right)\right] \tfrac{1}{3}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})[1 + O(g, \varepsilon)], \tag{A1}$$

where

$$C_{\mathrm{b}}^{(n)} = C_{\mathrm{b}}(3^{1/2}\pi^{-2/3}2^{-1/6})^{n-1} \exp\left[(n-1)\left(\frac{1}{3\varepsilon}+\frac{\zeta'(2)}{\pi^2}-\frac{1}{4}-\frac{\gamma}{2}\right)\right] \times \frac{3\pi^{n/2}}{n(n+2)\Gamma(n/2)}, \tag{A2}$$

should be compared with (2.1). The same manipulations as performed for the $n=1$ case give

$$g_{\mathrm{R}}(\mu) = g_{\mathrm{r}}(\mu) + \frac{\mathrm{i}C_{\mathrm{r}}^{(n)}}{\varepsilon}2^7 3\pi^4\left(-\frac{A}{g_{\mathrm{r}}(\mu)}\right)^{(d+n+2)/2} \exp\left(\frac{A}{g_{\mathrm{r}}(\mu)}\right)[1+\mathrm{O}(g_{\mathrm{r}})] \tag{A3}$$

with

$$g_{\mathrm{r}}(\mu) = g\mu^{-\varepsilon} - (n+8)g^2\mu^{-2\varepsilon}/8\pi^2\varepsilon + \mathrm{O}(g^3) \tag{A4}$$

and

$$C_{\mathrm{r}}^{(n)} = C_{\mathrm{b}}^{(n)} \exp[-(n+8)A/8\pi^2\varepsilon]. \tag{A5}$$

It is then straightforward to obtain the $\beta$ function

$$\beta(g_{\mathrm{R}}) = -\varepsilon g_{\mathrm{R}} + \frac{(n+8)}{8\pi^2}g_{\mathrm{R}}^2 + \mathrm{O}(g_{\mathrm{R}}^3) - \mathrm{i}C_{\mathrm{r}}^{(n)}2^7 3\pi^4\left(-\frac{A}{g_{\mathrm{R}}}\right)^{(d+n+4)/2} \exp\left(\frac{A}{g_{\mathrm{R}}}\right)[1+\mathrm{O}(g_{\mathrm{R}})]. \tag{A6}$$

This gives a non-trivial fixed point for negative $\varepsilon$

$$\underset{(\arg\varepsilon=\pi)}{g_{\mathrm{R}}^*} = \frac{8\pi^2\varepsilon}{n+8} + \mathrm{O}(\varepsilon^2) + \mathrm{i}C_{\mathrm{g}}^{(n)}\left(-\frac{(n+8)}{3\varepsilon}\right)^{(d+n+6)/2} \exp\left(\frac{n+8}{3\varepsilon}\right)[1+\mathrm{O}(\varepsilon)] \tag{A7}$$

where

$$C_{\mathrm{g}}^{(n)} = -2^{13/2}\pi\frac{9}{n+8}(3^{1/2}\pi^{-2/3}2^{-1/6})^{n-1}\frac{3\pi^{n/2}}{4\Gamma(\frac{1}{2}n+2)} \times \exp\left(\frac{(n+2)}{\pi^2}\zeta'(2) - \tfrac{1}{2}(n+6)\gamma - \tfrac{1}{4}(n+14) - \frac{(3n+14)}{(n+8)}\right). \tag{A8}$$

We will not give the renormalisation constants $Z^{\phi}$ and $Z^{\phi^2}$ for general $n$, since just as in the $n=1$ case only the perturbative parts of these constants give the leading high-order behaviour. Thus to this order

$$\eta(\varepsilon) = \gamma(g_{\mathrm{R}}^*) = [(n+2)/128\pi^4]g_{\mathrm{R}}^{*2}, \tag{A9}$$

$$-[\nu^{-1}(\varepsilon)-2] = \gamma_{\phi^2}(g_{\mathrm{R}}^*) = [(n+2)/8\pi^2]g_{\mathrm{R}}^*. \tag{A10}$$

From (A6)–(A10) it follows that

$$\underset{(\arg\varepsilon=\pi)}{\mathrm{Im}\,\eta(\varepsilon)} = C_{\eta}^{(n)}[-(n+8)/3\varepsilon]^{(d+n+4)/2}\exp[(n+8)/3\varepsilon][1+\mathrm{O}(\varepsilon)], \tag{A11}$$

$$\underset{(\arg\varepsilon=\pi)}{\mathrm{Im}[\nu^{-1}(\varepsilon)]} = C_{\nu^{-1}}^{(n)}[-(n+8)/3\varepsilon]^{(d+n+6)/2}\exp[(n+8)/3\varepsilon][1+\mathrm{O}(\varepsilon)], \tag{A12}$$

$$\underset{(\arg\varepsilon=\pi)}{\mathrm{Im}\,\omega(\varepsilon)} = C_{\omega}^{(n)}[-(n+8)/3\varepsilon]^{(d+n+8)/2}\exp[(n+8)/3][1+\mathrm{O}(\varepsilon)], \tag{A13}$$

where

$$C_{\eta}^{(n)} = -[(n+2)/24\pi^2]C_g^{(n)} \tag{A14}$$

and

$$C_{\nu^{-1}}^{(n)} = 3C_{\eta}^{(n)}, \qquad C_{\omega}^{(n)} = [3(n+8)/(n+2)]C_{\eta}^{(n)}. \tag{A15}$$

The above results for the imaginary parts of the critical exponents are of the form (4.4) and give, when substituted into (4.3), the required high-order behaviour as expressed by (4.6)–(4.10).

## References

Amit D J 1978 *Field Theory, the Renormalization Group and Critical Phenomena* (New York: McGraw-Hill)
Brézin E, Le Guillou J C and Zinn-Justin J 1977a *Phys. Rev.* D **15** 1544
—— 1977b *Phys. Rev.* D **15** 1558
Bruce A D and Wallace D J 1983 *J. Phys. A: Math. Gen.* **16** 1721
Collins J C and Soper D E 1978 *Ann. Phys.* **112** 209
Drummond I T and Shore G M 1979 *Ann. Phys.* **121** 204
Le Guillou J C and Zinn-Justin J 1977 *Phys. Rev. Lett.* **39** 95
—— 1980 *Phys. Rev.* B **21** 3976
Lipatov L N 1977a *Sov. Phys.–JETP Lett.* **25** 104
—— 1977b *Sov. Phys.–JETP* **45** 216
Lowe M J and Stone M 1978 *Nucl. Phys.* B **136** 177
McKane A J and Wallace D J 1978 *J. Phys. A: Math. Gen.* **11** 2285
't Hooft G 1977 *Proc. Summer School on Subnuclear Physics, Erice*
Wilson K G 1972 *Phys. Rev. Lett.* **28** 548
Zinn-Justin J 1973 *Cargèse lectures, unpublished*
—— 1982 in *Proc. Ecole d'Été, Les Houches 1982* ed J B Zuber (Amsterdam: North-Holland) to appear